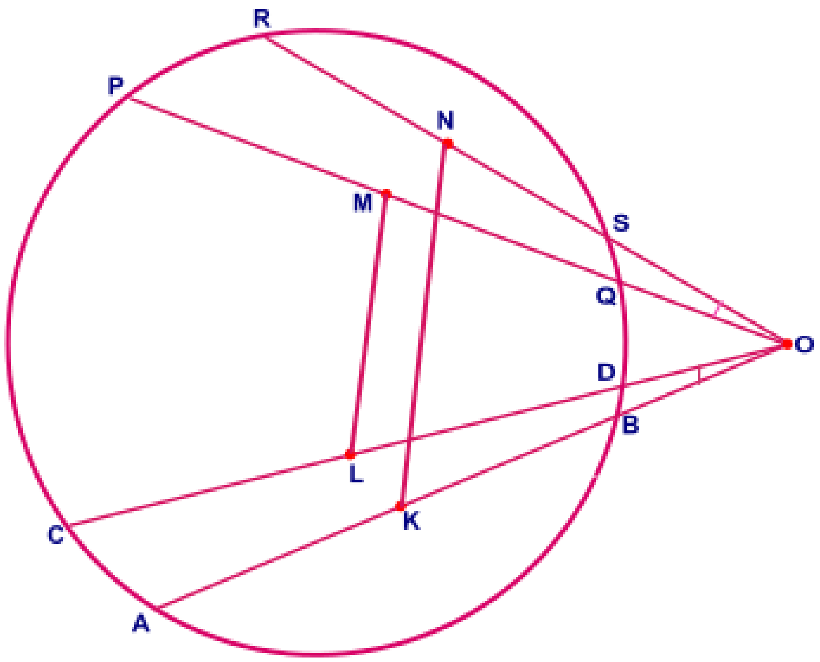
Author's Solution for Cash Award question of June 2023



In the above picture, O is a point outside the circle. From O, straight lines are drawn to the circle to create chords AB, CD, PQ & RS as shown in the picture. Given that $\angle AOC = \angle POR$ and K, L, M & N are the midpoints of AB, CD, PQ & RS respectively. Prove that LM ∥ KN.

Question framed by
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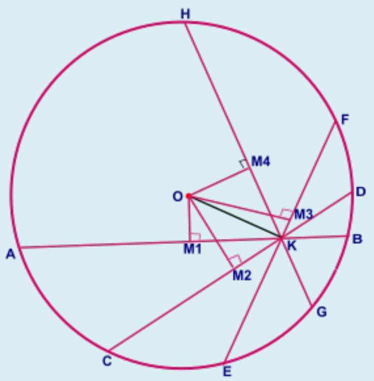
Author's Solution

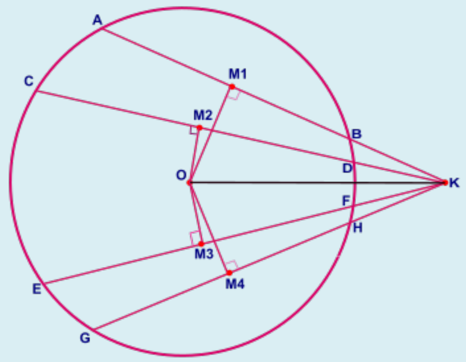
Before giving the solution let us see some geometric results which I am going to use in the solution.

Result -1:

If K is a point inside or outside of a circle with centre 'O' and if hundred or more chords of that circle pass through the said K, then, the midpoints of all those hundred or more chords will be concyclic in a single cirlce whose diameter is OK.

Proof for result -1





AB, CD, EF & GH are chords concurrent at 'K', a point outside the circle with centre 'O'

Construction :

Let M1, M2, M3 & M4 be the midpoints of chords AB, CD, EF & GF respectively.

Join OM1, OM2, OM3 & OM4. Join OK

Proof:

Since M1, M2, M3 & M4 are midpoints of the chords, OM1⊥ AB, OM2 ⊥ CD, OM3 ⊥ EF &

OM4 ⊥ GH (see the picture)

OK acts as hypotenuse for the right angled Δs OM1K, OM2K, OM3K & OM4K.

This shows that M1 & M2 lie on the semicircle formed on OK and M3 & M4 are also lying on the other semicircle formed on the same OK as diameter.

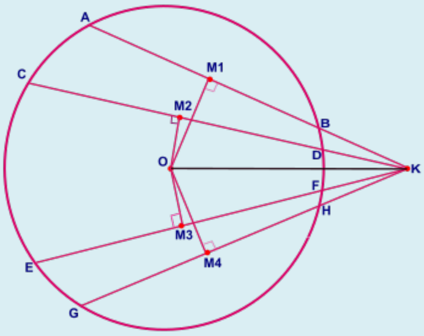
$\therefore$ $O, K,$ M1, M2, M3 & M4 are concyclic.

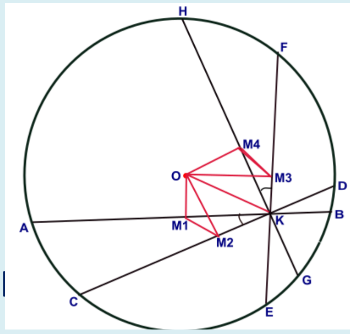
Similarly, the midpoint of any chord passing through 'K' will lie on this circle only.

------------- Proved.

Result - 2

When pairs of chords are concurrent on a point K inside or outside a circle with equal angles between them, then the distance between the midpoints of each such pair of chords is equal.





AB, CD, EF & GH are pairs of chords passing throug

$\angle AKC = \angle EKG$

To prove:

M1M2 = M3M4

Proof:

In the previous result, we proved that M1, M2, M3, M4 , O & K are concyclic. Now given that $\angle AKC = \angle EKG$**,**

ie $\angle M1KM2 = \angle M3KM4$.

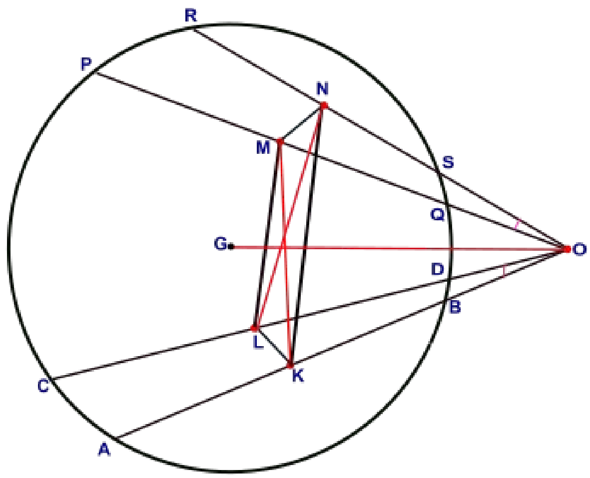
Since equal angles have equal chords,

M1M2 =M3M4

Similarly even if any other pair of chords with the same angle between them passes through K, the distance between their midpoints will also be equal to M1M2 or M3M4.

---------------Proved-------------

Now let us see the solution for the problem.



Given :

O is the point outside the circle.

chords AB, CD, PQ & RS are concurrent at O

$\angle AOC = \angle POQ.$

L, K, M & N are the midpoints of

AB, CD, PQ & RS respectively.

To prove:

LM || KN

Construction :

Mark the center of the circle G. Join GO, GK, GL, GM & GN, MN, LK, MK & LN.

Proof :

As per the result - 1 above (case 2) OGLMN are concyclic on a single circle.

As per result -2 chord LK = chord MN.

$\therefore \angle LMK = \angle MKN$

∴ LM || NK ------------------------- Proved.

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